Dirichlet's hyperbola method

Motivation: Estimate average for divisor function
$$\sum_{n\le x} \tau(n).$$

Apply convolution method: $\tau = \varepsilon * \varepsilon$

$$\sum_{n\le x} \tau(n) = \sum_{n\le x} \varepsilon * \varepsilon(n) = \sum_{n\le x} \sum_{d\mid n} 1$$

$$= \sum_{d\le x} \sum_{\substack{n\le x\\ n\equiv 0(d)}} 1 = \sum_{d\le x} \left\lfloor \frac{x}{d} \right\rfloor = \sum_{d\le x} \left(\frac{x}{d} + O(1) \right)$$

$$= x \sum_{d\le x} \frac{1}{d} + O(x).$$

Recall: $\sum_{d\le x} \frac{1}{d} = \log x + \gamma + O\left(\frac{1}{x}\right)$

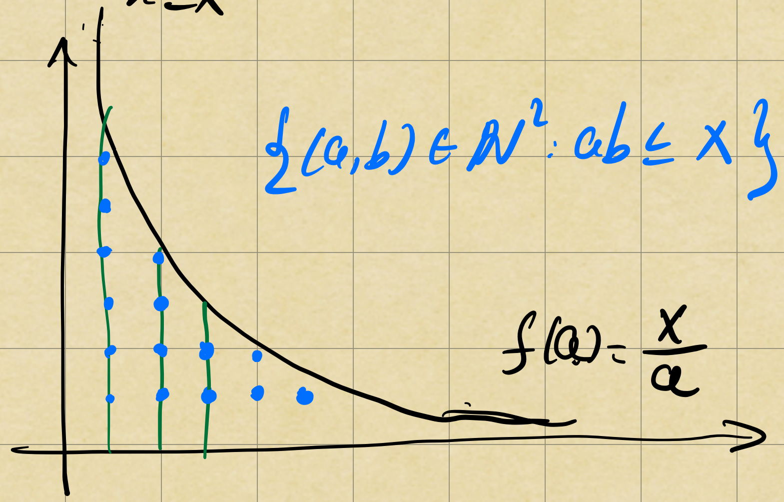
$$\Rightarrow \sum_{n\le x} \tau(n) = x \log x + O(x).$$

Can do better!

Theorem: $\sum_{n\le x} \tau(n) = x \log x + (2\gamma - 1)x + O(\sqrt{x}).$

Proof: Note that $\sum_{n\le x} \tau(n) = \sum_{n \le x} \sum_{ab=n} 1$

$$= \sum_{n\le x} |\{(a,b)\in \mathbb{N}^2 : ab=n\}| = |\{(a,b)\in\mathbb{N}^2 : ab\le x\}|.$$



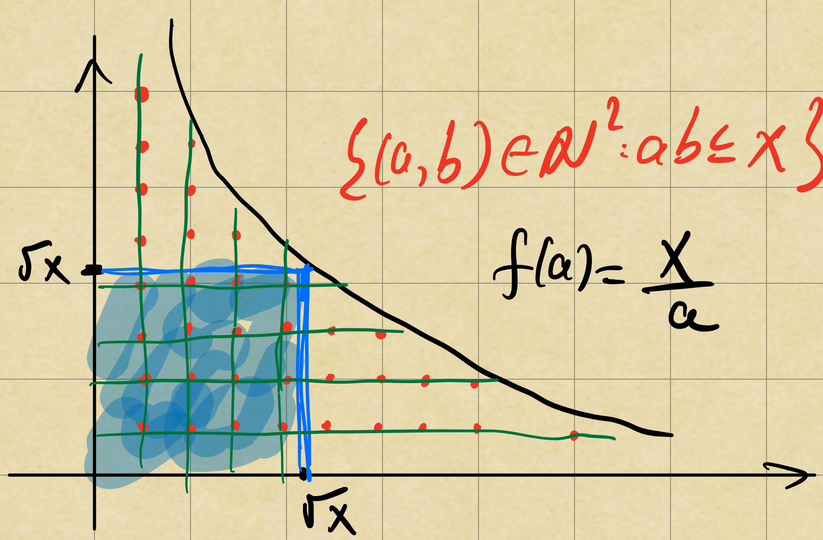
We count integer lattice points under the hyperbola!

Note: Using vertical counts

$$\sum_{n\le x}\tau(n) = \sum_{n\le x} |\{(n,b) : nb\le x\}|$$

$$= \sum_{n\le x} \left\lfloor \frac{x}{n} \right\rfloor = \sum_{n \le x}\left(\frac{x}{n} + O(1)\right) = x\lg x + O(x)$$

bad approximation, error term accumulates



Note: $ab \le x \implies a \le \sqrt{x}$ or $b \le \sqrt{x}$

Use vertical counts if $a \le \sqrt{x}$

horizontal counts if $b \le \sqrt{x}$

! Area $\{(a,b) \in \mathbb{N}^2 : a \le \sqrt{x}, b \le \sqrt{x}\}$ is double counted!

$$\sum_{n \le x} \tau(n) = \underbrace{\sum_{\substack{(a,b): ab \le x \\ a \le \sqrt{x}}} 1 + \sum_{\substack{(a,b): ab \le x \\ b \le \sqrt{x}}} 1}_{\text{they're equal by symmetry}} - \sum_{\substack{(a,b): ab \le x \\ a \le \sqrt{x}, b \le \sqrt{x}}} 1$$

$$= 2 \underbrace{\sum_{\substack{(a,b): ab \le x \\ a \le \sqrt{x}}} 1}_{(\mathrm{I})} - \underbrace{\sum_{\substack{(a,b): ab \le x \\ a \le \sqrt{x}, b \le \sqrt{x}}} 1}_{(\mathrm{II})}$$

$$(\mathrm{I}) = \sum_{a \le \sqrt{x}} \left\lfloor \frac{x}{a} \right\rfloor = \sum_{a \le \sqrt{x}} \left(\frac{x}{a} + O(1) \right)$$

$$= x\left(\log \sqrt{x} + \gamma + O\left(\frac{1}{\sqrt{x}}\right)\right) + O(\sqrt{x})$$

$$= \frac{1}{2} x \log x + \gamma x.$$

$$(\mathrm{II}) = \lfloor x \rfloor \cdot \lfloor x \rfloor = (\sqrt{x} + O(1))(\sqrt{x} + O(1))$$

$$= x + O(\sqrt{x})$$

$\boxtimes$

Dirichlet divisor problem: $\sum_{n\le x} \tau(n) = x\log x + (2\gamma - 1)x + \Delta(x)$.

We showed: $\Delta(x) \ll \sqrt{x}$. (Dirichlet 1849)

Classic bound: Voronoi (1904): $\Delta(x) \ll x^{1/3}\log x$

World record: Bourgain (2017): $\Delta(x) \ll_\varepsilon x^{\frac{517}{1648}+\varepsilon}$

$517/1648 = 0.3137\ldots$

Conjecture: $\Delta(x) \ll_\varepsilon x^{1/4+\varepsilon}$.

Elementary results about prime numbers

Definition: The von Mangoldt function $\Lambda(n)$ is

$$\Lambda(n) := \begin{cases} \log p, & \text{if } n = p^\ell,\ p \text{ prime},\ \ell \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases}$$

Lemma: $\Lambda = \log * \mu$.

Proof: Need to show $\Lambda(n) = \log * \mu(n)$, $\forall n \in \mathbb{N}$.

$n = 1$ ✓

Let $n = p_1^{l_1} \cdots p_k^{l_k}$, distinct primes p_i, $l_i \in \mathbb{N}$.

$$\Lambda * \mathcal{E}(n) = \sum_{d|n} \Lambda(d) = \sum_{i=1}^{k} \sum_{j=1}^{l_i} \Lambda(p_i^k)$$

$$= \sum_{i=1}^{k} l_i \log(p_i) = \log(p_1^{l_1} \cdots p_k^{l_k}) = \log n.$$

__Definition:__ $\psi(x) := \sum_{n \le x} \Lambda(n)$

$$\theta(x) := \sum_{n \le x} \mathbb{1}_p(n)\Lambda(n) = \sum_{p \le x} \log p.$$

__Lemma:__ $\psi(x) = \theta(x) + O(\sqrt{x}(\log x))$.

__Proof:__

$$0 \le \psi(x) - \theta(x) = \sum_{\substack{p \text{ prime} \\ m \ge 2,\, p^m \le x}} \log p$$

$$= \sum_{p \le \sqrt{x}} \log p \left\lfloor \frac{\log x}{\log p} \right\rfloor \rightsquigarrow p^m \le x \Rightarrow m \le \frac{\log x}{\log p}$$

$\rightsquigarrow p^m \le x$, for $m \ge 2 \Rightarrow p \le \sqrt{x}$

$$\le \sum_{p \le \sqrt{x}} \cancel{\log p} \cdot \frac{\log x}{\cancel{\log p}} \le \sqrt{x} \log x. \quad \blacksquare$$

__Theorem:__ The following are equivalent:

(i) $\pi(x) \sim \dfrac{x}{\log x}$ (PNT)

(ii) $\psi(x) \sim x$

(iii) $\theta(x) \sim x$

__Proof:__ We've already seen (ii) $\Leftrightarrow$ (iii).

"(i) $\Rightarrow$ (iii)" Assume $\pi(x) \sim \frac{x}{\log x}$.

By Abel summation ($f = 1\!\!1_P$, $g = \log$)

$$\sum_{p \le x} \log p = \pi(x) \log x - \int_1^x \frac{\pi(t)}{t}\, dt$$

$\pi(t) = 0$ for $t < 2$

$$= \left(\frac{x}{\log x} + o\left(\frac{x}{\log x}\right)\right) \log x - \int_2^x \frac{\pi(t)}{t}\, dt$$

Note $\int_2^x \frac{\pi(t)}{t} dt \ll \int_2^x \frac{1}{\log t} dt = O\left(\frac{x}{\log x}\right)$.

x "large"

$= o(x)$

$\Longrightarrow \theta(x) = x + o(x)$ ✓

"(iii) $\Rightarrow$ (i)" By Abel summation:

$$\pi(x) = \sum_{n \le x} 1\!\!1_P(n) \log n \cdot \frac{1}{\log n}$$

$$= \frac{\theta(x)}{\log x} + \int_{1.5}^x \frac{\theta(t)}{t (\log t)^2}\, dt$$

$$= \frac{x}{\log x} + o\left(\frac{x}{\log x}\right) + O\left(\int_{1.5}^x \frac{1}{(\log t)^2} dt \right)$$

$$\int_{1.5}^{x} \frac{1}{(\log t)^2}\, dt = \int_{1.5}^{\sqrt{x}} \frac{1}{(\log t)^2}\, dt + \int_{\sqrt{x}}^{x} \frac{1}{(\log t)^2}\, dt$$

$$\ll \sqrt{x} + \frac{x}{(\log \sqrt{x})^2} \ll \frac{x}{(\log x)^2}. \qquad \boxtimes$$

<u>Theorem</u> (Chebyshev) We have that

$$\pi(x) \asymp \frac{x}{\log x}, \quad \theta(x) \asymp x, \quad \psi(x) \asymp x.$$

<u>Proof</u>: Let $S(x) := \sum_{n \le x} \log n - 2 \sum_{n \le \frac{x}{2}} \log n$.

Recall $\sum_{n \le x} \log n = x \log x - x + O(\log x)$

(since log increasing)

$$\Rightarrow S(x) = (\log 2) \cdot x + O(\log x).$$

Now we use $\log = \Lambda * \mathcal{E}$, so

$$\sum_{n \le x} \log n = \sum_{d \le x} \Lambda(d) \sum_{n \le \frac{x}{d}} 1 = \sum_{d \le x} \Lambda(d) \left\lfloor \frac{x}{d} \right\rfloor$$

$$\Rightarrow S(x) = \sum_{d \le x} \Lambda(d) \left\lfloor \frac{x}{d} \right\rfloor - 2 \sum_{d \le \frac{x}{2}} \Lambda(d) \left\lfloor \frac{x}{2d} \right\rfloor$$

$$= \sum_{d \le x} \Lambda(d) \left(\left\lfloor \frac{x}{d} \right\rfloor - 2 \left\lfloor \frac{x}{2d} \right\rfloor \right)$$

(Note $\left\lfloor \frac{x}{2d} \right\rfloor = 0$ if $d > \frac{x}{2}$).

Note: For all $\alpha \geq 1$, $\lfloor \alpha \rfloor - 2\lfloor \frac{\alpha}{2} \rfloor \in \{0, 1\}$.
(exercise).

This implies $\psi(x) - \psi(\frac{x}{2}) \leq S(x) \leq \psi(x)$ $(*)$

because $\lfloor \frac{x}{d} \rfloor - 2\lfloor \frac{x}{2d} \rfloor = 1$ if $\frac{x}{2} < d \leq x$.

This implies $\psi(x) \geq S(x) \gg x$. ✓

For the other direction, we have $\psi(x) - \psi(\frac{x}{2}) = O(x)$.

$$\psi(x) = \psi(x) - \psi\left(\tfrac{x}{2}\right) + \psi\left(\tfrac{x}{2}\right) - \psi\left(\tfrac{x}{4}\right) + \dots + \underbrace{\psi\left(\tfrac{x}{2^j}\right) - \psi\left(\tfrac{x}{2^{j+1}}\right)}_{\text{eventually } 0}$$

$$= \sum_{j=0}^{\infty} \left(\psi\left(\frac{x}{2^j}\right) - \psi\left(\frac{x}{2^{j+1}}\right) \right)$$

$$= \sum_{j=0}^{\infty} O\left(\frac{x}{2^j}\right) = O(x).$$

This proves $\psi(x) \asymp x$, and hence $\theta(x) \asymp x$.

To deduce $\pi(x) \asymp \frac{x}{\log x}$, first note

$$\pi(x) = \sum_{p \leq x} 1 \geq \frac{1}{\log x} \sum_{p \leq x} \log p = \frac{1}{\log x} \theta(x) \gg \frac{x}{\log x}.$$

For the other direction,

$$\pi(x) \leq \pi(\sqrt{x}) + \sum_{\sqrt{x} < p \leq x} \frac{\log p}{\log \sqrt{x}} \leq \sqrt{x} + 2\frac{\theta(x)}{\log x} \ll \frac{x}{\log x}.$$

$\boxtimes$

Mertens' theorems

We cannot count primes precisely (yet!), but we can obtain useful results if we add some "weights".

Theorem: $\displaystyle\sum_{p \leq x} \frac{\log p}{p} = \log x + O(1).$

Proof: **Step 1**: $\displaystyle\sum_{d \leq x} \frac{\Lambda(d)}{d} = \log x + O(1).$

Proof of claim: Note that since $\log = \Lambda * \varepsilon$,

$$\sum_{n \leq x} \log n = \sum_{n \leq x} \Lambda * \varepsilon(n) = \sum_{d \leq x} \Lambda(d) \left\lfloor \frac{x}{d} \right\rfloor$$

$$= x \sum_{d \leq x} \frac{\Lambda(d)}{d} + \underbrace{O(\psi(x))}_{= O(x)}$$

But $\displaystyle\sum_{n \leq x} \log n = x \log x + O(x).$

$$\rightarrow x \sum_{d \leq x} \frac{\Lambda(d)}{d} = x \log x + O(x) \checkmark$$

Step 2: $\displaystyle\sum_{p\le x}\frac{\log p}{p}=\sum_{d\le x}\frac{\Lambda(d)}{d}+O(1).$

$$\left|\sum_{p\le x}\frac{\log p}{p}-\sum_{d\le x}\frac{\Lambda(d)}{d}\right|=\sum_{p\le x^{1/2}}\log p\sum_{2\le m\le\frac{\log x}{\log p}}\frac{1}{p^k}$$

$$\le\sum_{p\le x^{1/2}}\frac{\log p}{p^2}\underbrace{\left(\frac{p}{p-1}\right)}_{\le 2}\ll\sum_{p}\frac{\log p}{p^2}$$

$$\ll\sum_{n}\frac{\log n}{n^2}\ll 1.$$

exercise: ⊛

⊛ $\displaystyle\sum_{n}\frac{\log n}{n^2}\ll\sum_{n\ge 2}n^{-3/2}\ll\int_1^\infty t^{-3/2}\,dt=O(1)$

Theorem: $\exists$ an absolute constant M such that, for $x\ge 3$, we have

$$\sum_{p\le x}\frac{1}{p}=\log\log x+M+O\left(\frac{1}{\log x}\right).$$

Proof: We deduce it from previous Mertens thm by partial summation.

Let $\displaystyle S(x)=\sum_{p\le x}\frac{\log p}{p}=\log x+O(1).$

Hence $S(x) = \log x + E(x)$, where $E(x) \ll 1$.

$$\sum_{p \le x} \frac{1}{p} = \frac{S(x)}{\log x} + \int_{3/2}^{x} \frac{S(t)}{(\log t)^2 \cdot t}\, dt$$

$$= \frac{\log x}{\log x} + \frac{E(x)}{x} + \int_{3/2}^{x} \frac{1}{\log t \cdot t}\, dt + \int_{3/2}^{x} \frac{E(t)}{(\log t)^2 t}\, dt$$

$$= \log\log x + \underbrace{\left(\int_{3/2}^{\infty} \frac{E(t)}{(\log t)^2 t}\, dt + 1 - \log\log\frac{3}{2} \right)}_{M}$$

$$+ O\Big(\frac{1}{x} + \underbrace{\int_{x}^{\infty} \frac{1}{t(\log t)^2}\, dt}_{\le \frac{1}{\log x}} \Big)$$

$$= \log\log x + M + O\left(\frac{1}{\log x}\right).$$

$\square$

Theorem: $\exists A > 0$ constant such that for $x \ge 2$

$$\prod_{p \le x} \left(1 - \frac{1}{p}\right) = \frac{A}{\log x} + O\left(\frac{1}{(\log x)^2}\right).$$

Proof: From Taylor expansion, for $y \in (-1,1)$,

$$\log(1-y) = -\sum_{n=1}^{\infty} \frac{y^n}{n}.$$

Thus

$$\sum_{p \le x} \log\left(1-\frac{1}{p}\right) = -\sum_{p\le x}\frac{1}{p} - \sum_{p \le x}\sum_{n=2}^{\infty}\frac{1}{np^n}$$

$$= -\sum_{p\le x}\frac{1}{p} - \underbrace{\sum_{p}\sum_{n=2}^{\infty}\frac{1}{np^n}}_{B.} + O\Big(\underbrace{\sum_{p>x}\sum_{n=2}^{\infty}\frac{1}{np^n}}_{\ll \frac{1}{x}}\Big)$$

Note

$$\sum_{p>x}\sum_{n=2}^{\infty}\frac{1}{np^n} \le \sum_{p>x}\frac{1}{p^2}\sum_{n=0}^{\infty}\frac{1}{p^n}$$

$$\le \sum_{p>x}\frac{1}{p(p-1)} \le \sum_{n>x}\frac{1}{n(n-1)} \ll \frac{1}{x}.$$

Therefore

$$\sum_{p\le x}\log\left(1-\frac{1}{p}\right) = -\log\log x + (M+B) + O\left(\frac{1}{\log x}\right).$$

We use that for $|y|<1$, $\exp(y) = 1 + O(y)$.

Therefore

$$\exp\left(O\left(\frac{1}{\log x}\right)\right) = 1 + O\left(\frac{1}{\log x}\right).$$

$$\Rightarrow \prod_{p\le x}\left(1-\frac{1}{p}\right) = \frac{e^{M+B}}{\log x}\left(1+O\left(\frac{1}{\log x}\right)\right) \quad \square$$